Mathematical Understanding and “What if things had been different?” Questions

Andrei Mărăşoiu

According to Grimm (2014), we only understand a phenomenon if we know what other phenomena it depends on, and we identify dependencies according to how we answer “What if things had been different?” questions. I argue that this view meets with mathematical counterexamples. For, in mathematics, things couldn't have been different. I consider three replies Grimm may make, and argue they do not succeed.

Introduction

On Grimm’s (2014) view, presented in Section 1, we only understand a phenomenon if we know what that phenomenon depends on. We understand how things stand because we know how they could have differed. That is, we know what depends on what because we’re able to answer questions of the form “What if things had been different?”

However, in Section 2, I mention two instances of mathematical understanding that, I argue, are counterexamples to Grimm's view of understanding. The first is understanding the conclusion of a simple arithmetical argument on the basis of its premises. The second is understanding an axiom of set theory, the axiom of infinity. I argue that understanding this axiom, and understanding the arithmetical truths in the argument given, are not mediated by our ability to answer “What if things had been different?” questions.

In Sections 3-5, respectively, I consider three replies Grimm may make, which interpret

“What if things had been different?” questions differently: a purely modal approach, a grounding approach, and a modeling approach to answering such questions. I will argue that none of these approaches leads to re-interpreting the examples given in Section 2 so that they no longer undermine Grimm’s view.

The Conclusion explores the wider epistemological consequences of this dialectical situation.

1. Understanding as knowledge of dependencies

In this section, I summarize Grimm’s view of understanding.[1] He writes:

> Since understanding seems to arise from a grasp of all these different types of dependence, it might therefore be better [to] claim that understanding consists… of something like “knowledge of dependency relations” (Grimm 2014, p. 341).

1 This view is in the background of much contemporary theorizing of understanding, such as Zagzebski's (2001) virtue epistemology of understanding, and Gopnik's (1998) view of understanding as knowing the causal structure of the phenomenon understood. Closest to Grimm seems to be Greco’s view, for whom: “[T]o have understanding is to have systematic knowledge of dependence relations. To understand a thing is to be able to (knowledgeably) locate it in a system of appropriate dependence relations” (Greco 2014, p. 286).

Consider an example, borrowed from Carr (1961). Suppose you set out to understand the 1905 Revolution (a social unrest brutally repressed by Czarist authorities in St. Petersburg in 1905). The occurrence of the 1905 Revolution depended on sundry factors: macro-economical, social, mass-psychological. Unions were prohibited and peasants newly arrived in the city had no workplace protections. Previous protests had been met with violence from riot police, and unrest was growing. Workers wanted acknowledgment of their problems. The newly created social stratum of industrial workers was disadvantaged in terms of income and social status. Our understanding of the 1905 Revolution is partly constituted by our knowledge of the variety of its causes, as well as their interrelations. Efficient causation is only one kind of relevant dependence. Other kinds of dependence are temporal or logical relations (e.g., the 1905 Revolution preceded the 1917 Revolution; and it instantiates the concept of revolution). The 1905 Revolution *depends on* each of these because the Revolution might have not occurred at all, or might have occurred differently than it did, had any of these factors been absent or modified. Understanding is a matter of degree: we know more or fewer factors the Revolution depended on; and we know, in more or less depth, how each of them influenced the Revolution's unfolding.

The view that understanding requires knowing what the phenomenon understood[2] depends on flows from other of Grimm's commitments. Grimm (2010, p. 337) argues that "understanding is the goal of explanation," so that at the end of explanatory inquiry the

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[2] The word "phenomenon" is used broadly: any state of affairs, event, process unfolding, or property obtaining count as phenomena.

epistemic agent is able to explain why the phenomenon understood occurred. Moreover, Grimm (2014, p. 341) adheres to Kim's view that "explanations track dependence relations" (1994, p. 183).[3] And we identify what a phenomenon understood depends on by answering "What if things had been different?" questions:

> Following James Woodward (2003), one promising way to spell out the requisite notion of the kind of achievement that is needed—hence the kind of grasping that is involved in understanding—is in terms of having an ability to answer ''what-if-things-had-been-different?'' questions. To have an ability to answer questions of this sort, Woodward argues, is to be able to anticipate the sort of change that would result in the thing we want to explain (the explanandum) if the factors cited as explanatory (the explanans) were different in various ways. Brian Skyrms… expresses this same point in a more metaphorical way. According to

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3 As Kim elaborates: "[M]y claim will be that dependence relations of various kinds serve as objective correlates of explanations. Dependence, as I will use the notion here, is a relation between individual states and events; however, it can also relate facts, properties, regularities between events, and even entities. We speak of the "causal dependence" of one event or state on another; that is one type of dependence, obviously of central importance. Another dependence relation, orthogonal to causal dependence and equally central to our scheme of things, is mereological dependence: ...the properties of a whole, or the fact that the whole instantiates a certain property, may depend on the properties had by its parts" (Kim, 1994, p. 183).

> Skyrms, grasping involves having an ability to anticipate that ‘‘wiggling’’ one variable will characteristically lead to a ‘‘wiggling’’ of another variable. (Grimm, 2006, pp. 532-533).

For instance, if fewer poor people had moved from the countryside to the city; or if the government had instructed the riot police to react with caution to occasional conflicts on the streets and in factories; or if better workplace protections had been secured; or if the workers had been allowed to unionize and start negotiating their position (etc.), then perhaps the 1905 Revolution might have been less bloody than it was, or might have occurred later than it did.

We can put Grimm's view of understanding as follows:

(1) If we understand a phenomenon, then we know how to explain why it occurs.

(2) If we know how to explain why a target phenomenon occurs, then we know what other phenomena the target phenomenon depends on.

(3) If we know what other phenomena the target phenomenon depends on, then we are able to answer “What if things had been different?” questions about those phenomena.

(4) So, if we understand a target phenomenon, then we are able to answer “What if things had been different?” questions about what the target phenomenon depends on.

Or, as Grimm puts it, understanding involves “an apprehension of how things stand in modal space” (2014, p. 334). The argument is valid. Presumptive counterexamples may be produced, which challenge the notion that understanding requires knowledge, or being able to explain what one understands.[4] But I am interested in what happens when everything goes well, epistemically speaking. So I will grant (1) for the sake of argument. Similarly, one may advance alternative conceptions of explanation, or resist any general view about scientific explanation; but I will grant (2) for the sake of argument.

I will argue in the next section that instances of mathematical understanding provide counterexamples to Grimm's view[5] because, in mathematical cases, one *cannot* go about answering the question "Why is that the case?" by answering the question "What if things had been different?" For, in mathematical cases, things *couldn't* have been different. And considering patent impossibilities seems explanatorily idle, rather than conducive to understanding.

As I granted (1) and (2) of Grimm's view, and since, in the next section, I offer two counterexamples to (4), I reject (3): the kind of understanding we have of phenomena we can explain doesn't always require that we are able to answer "What if things had been different?" questions. In Sections 3-5, respectively, I consider three replies Grimm may make, intended to re-interpret the instances of mathematical understanding I mention so as to ensure they do not conflict with Grimm's view. I will argue that the replies fail to achieve that end.

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4 For discussion surrounding whether understanding a phenomenon requires being able to explain it, see Lipton (2009), Khalifa (2012) and Strevens (2013).

5 Grimm's extending the "knowledge of causes" view to mathematical understanding contrasts with Woodward's (2003, p. 3) explicit disavowal of applying his view of scientific explanation to mathematical cases.

2. Two simple cases of mathematical understanding

In this section, I argue that two instances of mathematical understanding[6] are counterexamples to Grimm's view.

Initially, Grimm (2006) had advanced a conception of understanding as “knowledge of causes,” according to which we only understand that which we know the causes of. However, this conception involved efficient causation alone, and couldn't be generalized to cover mathematical understanding. To remedy that, Grimm (2014) advocates a theory of understanding based on dependencies in general, rather than on efficient causation alone:

> given how closely our notion of causation is tied to pushing and pulling – to exerting causal force – a perhaps more attractive strategy would be to demote the notion of causation from its central role and instead to appeal more generally to the notion of dependence (Grimm, 2014, p. 341).

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6 The discussion to follow doesn't presuppose that mathematical understanding is a unified category of epistemic evaluation. There are a variety of mathematical projects worth pursuing, and they largely differ in their characteristic epistemic features. See Avigad (2008) and Hafner and Mancosu (2005) for different kinds of explanation in mathematics, and, presumably, different kinds of attendant understanding.

There are a number of problems with this passage that I bracket.[7] However, I will argue that the resulting view faces counterexamples.

For the first counterexample, consider the understanding provided by the following argument:

(a) Every strictly positive natural number has a predecessor.

3 is a strictly positive natural number.

So, 3 has a predecessor.

The argument is valid. The premises are true, so is the conclusion. Moreover, the premises jointly *explain* the conclusion. One way of *understanding* the conclusion – the instance of understanding I'll discuss – is based on the premises. I will argue that Grimm's account of the understanding we have of the conclusion on the basis of the premises in (a) is false.

On Grimm's view, an answer to the question "Why are things so?" that provides an understanding of the state of affairs explained is obtained from an answer to the question "What if things have been different?" Given that the premises in (a) jointly provide an understanding of the conclusion in (a), that understanding has to be obtained *via* (b) below, since (b) answers the "What if things had been different?" question with respect to (a):

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[7] To consider only one issue: Our ordinary concept of causation *isn't* push-pull. We think demand may generate supply, Congress can keep the President in check, and a word can hurt our feelings; none of these are push-pull, and all are causal according to common sense.

(b) If 3 had not been a strictly positive natural number, or if not every strictly positive natural number had a predecessor, then 3 might have not had a predecessor.

Statement (b) is a mathematical counterfactual. How to interpret mathematical counterfactuals is too large an issue to settle here.[8] It's clear that, when we understand why 3 has a predecessor on the basis of the facts that every strictly positive natural number has a predecessor, and 3 is such a number, what we are entertaining is (a), not (b). We have a grasp of the relevant dependencies (3 is a strictly positive natural number, and hence shares their properties) without any appeal to counterfactuals like (b).

Here is a second example: the set-theoretic axiom of infinity, according to which there are infinitely many things. This claim is necessarily true if true at all, for there doesn't seem to be any room to ask what the existence of infinitely many things might be *contingent* on.[9]

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I submit it would be mistaken to think all countermathematicals are trivially true given falsity of antecedents. Consider: “If one [says] ‘nothing sensible can be said about how things would be different if the axiom of choice were false,’ it seems wrong…: if the axiom of choice were false, the cardinals wouldn’t be linearly ordered, the Banach-Tarski theorem would fail and so forth” (Field, 1989, pp. 237-238). Field is right. To appreciate whether the conditional is true, we have to grasp a bit of set theory and topology. That isn't trivial, regardless of whether the Axiom of Choice is true or false.

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In the text, I only need the claim that the axiom of infinity is necessarily true if true at all.

Suppose, for the sake of argument, that the axiom of infinity is true. It follows that no possible worlds exist where it is false. It's hard to see how the mere inexistence of possible worlds falsifying the axiom would *explain* the axiom or help us understand it.[10]

I sketched two instances of mathematical understanding: the understanding afforded, *via* a simple arithmetical argument, of the conclusion by the premises; and the understanding we have of an axiom of set theory. These instances of understanding are importantly different. The first is within the grasp of grade school children; the second is more foundational. The first is arithmetical, the second is set-theoretical. The first is proof-based; the second is an axiom. If, as I contend, both are counterexamples to Grimm's view, this shows that the problems with that view are quite general.

In Sections 3-5, respectively, I'll consider three replies Grimm might make, in which the instances of mathematical understanding I just mentioned aren't counterexamples to his view. I will argue that these replies don't succeed in explaining away the counterexamples. As Grimm's view of understanding was meant to apply to all cases of scientific understanding – mathematics

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There are traditional ways to deny this claim (such as Mill's suggestion that arithmetical truths are inductive generalizations generated by counting pebbles), but they seem to me to lack any plausibility.

[10] In keeping with (1), one may wonder in what sense we may explain the axiom of infinity when we understand it. Maddy (2011, pp. 113-137) addresses the issue, considering a form of explanation specific to foundational axioms, which she terms "elucidation." How elucidation may differ from other forms of understanding is beyond the scope of this text.

included – it should be abandoned, given the compelling counterexamples it faces.

3. A modal approach to answering "What if things had been different?" questions

In this section, I argue against a reply that Grimm could make countering the notion that the instances of mathematical understanding mentioned in Section 2 are counterexamples to his view. The reply is that we *do* have an answer to "What if things had been different?" questions. Namely, that they couldn't have been different. This answer is satisfactory with respect to the examples given in Section 2 because knowing how things *must* stand in the mathematical realm leads to an understanding of why things actually are that way.

I believe this reply misinterprets "What if things had been different?" questions. Consider a mundane example. In preparation for surgery, a patient is given an anesthetic. The dosage has to be right if the operation is to proceed seamlessly. We may ask: what if things had been different? Had the anesthetic dose been smaller, the patient might have woken up on the operating table. Had the anesthetic dose been much larger, the patient could have had a heart attack. By intervening on the anesthetic dose, we see effects in the patient's health condition. We know why the patient was sound asleep during surgery but woke up fine afterwards: it was partly because he had been administered the right dose of anesthetic. Intervening on causes often alters their effects, so answering the question "What if things had been different?" in this context leads to the discovery of causal relations.

By varying the anesthetic dose, we vary its effects. In contrast, in mathematics, there's nothing to vary, so long as we agree with Grimm's (2014, p. 334) claim that knowledge of mathematical truths is *a priori*. So "What if things had been different?" questions clearly must play a different role in understanding mathematical matters than in understanding others, e.g., how anesthetics work.[11]

Once we admit that mathematical statements are necessarily true if true at all, then answering questions of the form "What if things had been different?" is trivialized. Compare "If every natural number has a successor, then 1+1 exists" with "If infinite sets exist, then 1+1 exists." If both are necessarily true, then a purely modal approach will be unable to ascertain that the first expresses a genuine dependence whereas the second does not. Assuming that we do understand that 3 has a predecessor because all strictly positive natural numbers do, then either that understanding is not given *via* an explanation such as that in (a), *contra* (1), or that explanation doesn't refer to dependencies between arithmetical facts, *contra* (2). If a purely modal approach to "What if things had been different?" questions in arithmetic trivializes their answers, given that banalities explain nothing, it is hard to see how a purely modal approach could underwrite our understanding of arithmetic.

Moreover, if answers to "What if things had been different?" in mathematics are

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11 We should distinguish Grimm's view from a related one. "One way in which our wonderment about a phenomenon can be relieved is through a demonstration that it is necessary, that it could not be otherwise" (Glymour, 1980, p. 32). Sometimes truly understanding why something *actually* obtains requires knowing that it obtains *necessarily*. This is surely true sometimes, and may well apply often in mathematical matters. But it seems to obviate an appeal to the "What if things had been different?" question, rather than require it. Glymour is clear that this kind of understanding by knowing-it-is-necessary only holds *sometimes*.

trivialized, then finding them is no epistemic achievement. Even though we may readily produce such answers, it is not thanks to the exercise of our cognitive abilities.. Grimm endorses (1), by which we only understand what we know how to explain. It follows that an element of *cognitive achievement* corresponds to coming to understand something new, and what is achieved is learning what the phenomena understood depend on. This shows that, if (3) is assumed, (1) is undermined.

One might reply that counterfactuals like (b) can be interpreted so that they are non-trivial, by considering not only metaphysically possible worlds, but also metaphysically impossible worlds in which mathematical truths could possibly be false (Nolan, 2014). Perhaps such a reply succeeds in making the attribution of a truth value to (b) non-trivial. But it doesn't alter the fact that (b) isn't required in order to understand the conclusion of (a) – that 3 has a predecessor – based on its premises: that all strictly positive natural numbers have predecessors, and 3 is such a number. And it is hard to see how an approach that considers metaphysically impossible worlds would *illuminate*, or enhance our understanding of the axiom of infinity.

In the next two sections, I explore whether two other ways of interpreting "What if things had been different?" questions are conducive to explaining and understanding phenomena inquired into. I will argue that this is not always so in the mathematical realm.

4. Grounding to the rescue?

Grimm mentions grounding (2014, p. 341),[12] suggesting the possibility of elaborating a

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reply to the claim that Grimm's view faces counterexamples relying on a grounding-based metaphysics. In this section, I argue that appeal to grounding is ill-suited to re-interpreting the instances of mathematical understanding mentioned in Section 2.

The idea of appealing to grounding in order to fine-grain a modal approach to the dependencies grasped in understanding a phenomenon seems to be the following. To know what a phenomenon is caused by (*lato sensu)*, or depends on, when we understand that phenomenon, is to know the grounds of that phenomenon. The idea that explaining why the phenomenon understood occurs is accounted for by answering the question "What if things had been different?" takes on a new guise: vary the grounds, and you will have thereby varied the grounded. By building a profile of how the grounds of a phenomenon vary, one can explain why the phenomenon occurs as it does.

In fact, Grimm mentions Aristotelian formal causation, as well. He regards the choice between appeal to grounding and appeal to formal causation as alternative accounts of metaphysical dependence in general as terminological (a diagnosis I don't share). Grimm states that "Aristotle's notion of causation was more expansive, along the lines developed here," (2014, p. 341) referring to Aristotelian causes, in their formal, material, efficient, and final varieties. What formal causes are is quite obscure, but that problem can be side-stepped because these varieties of causes easily generalize to the plurality of dependencies. Mancosu (1996) chronicles how geometers gradually gave up talking about formal causes as they began articulating a notion of demonstrative proof anticipating those current nowadays. This is why I focus on grounding as a way to articulate Grimm's view of how understanding involves knowledge of what grounds what.

Wilson argues that a general notion of grounding isn't theoretically helpful in illuminating what metaphysical dependencies are:

> The problem here is not just that claims of Grounding (failure of Grounding) leave open some interesting questions; it is that such claims admit of such underdetermination —about whether the dependent goings-on exist, are reducible or rather distinct from the base goings-on, are efficacious, and so on—that even basic assessment of claims of metaphysical dependence, or associated views, cannot proceed by reference to Grounding alone. As such, investigations into metaphysical dependence cannot avoid appealing to the specific 'small-g' grounding relations (again: type or token identity, functional realization, the classical mereological part–whole relation, the causal composition relation, the set membership relation, the proper subset relation, the determinable–determinate relation, etc.) that are capable of answering these crucially basic questions about the existential, ontological, metaphysical, and causal status of metaphysically dependent goings-on (Wilson 2014, p. 540).

Wilson's point is the following. The theoretical construct of Grounding (capitalized to distinguish it from specific dependence relations) is supposed to capture metaphysical dependence in general. But this theoretical construct can't by itself help answer specific questions: "whether the dependent goings-on exist, are reducible or rather distinct from the base goings-on, are efficacious, and so on." These questions, however, need answering. This is why one should look for another way to describe various metaphysical dependencies; and Wilson argues that part-whole, logical, causal relations etc. provide satisfactory answers to the questions that need answering. I endorse Wilson's criticism, and think it is the advocate of Grounding who has the burden of proof to show that Grounding can provide an illuminating account of metaphysical dependence.

How might a "small-g" grounding version of Grimm's view address the counterexamples in Section 2? Take the arithmetical counterexample first. The fact that every strictly positive natural number has a predecessor, and the fact that 3 is a strictly positive natural number, jointly *ground* the fact that 3 has a predecessor. This is an improvement over a purely modal approach, for now there are part-whole and logical relations that can be invoked as relations of metaphysical dependence going beyond whether the mathematical facts in question hold with metaphysical necessity or not. Because 3 is *a part of* the strictly positive natural numbers, the fact that every strictly positive natural number has a property (having a predecessor) *logically implies* that 3 has that property too. These logical and part-whole relations can do justice to the understanding-giving and explanatory asymmetry of the premises and conclusion of (a), improving over a purely modal elaboration of Grimm's view.

However, a grounding elaboration of Grimm's view has trouble accounting for (b). This is because (b) is an answer to "What if things had been different?", whereas grounding relates to metaphysical necessity, and it *doesn't* seem to make sense to ask what it might have been for metaphysically necessary facts (that every strictly positive natural number has a predecessor, that 3 is such a number, etc.) to be different from what they are. We lack an answer to the question of what it is to *vary* mathematical grounds. By the same token, we can't vary facts (taken to be metaphysically necessary) such as the existence of infinitely many things.[13] So a grounding-

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based reply in support of Grimm's view doesn't succeed in addressing the understanding we have of the axiom of infinity either.

I have considered an interpretation of “What if things had been different?” questions on which answering them indicates what grounds what, for varying the grounds would vary the grounded. And I have argued that, while this improves over a purely modal approach to “What if things had been different?” questions, it doesn’t rescue Grimm’s view from counterexamples.

5. Different models

Grimm (2016) argues that “mental models” play an important role in the cognitive activity that underwrites understanding. This raises the question of whether a modeling approach to understanding[14] could rescue Grimm’s view from counterexamples. The guiding thought

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One might propose that the existence of infinitely many things is grounded in the continuity of spacetime. However, that spacetime is continuous (if it is) seems to be a claim of less modal strength than what the axiom of infinity requires, for it is a claim in the foundations of physics, not of set theory.

14 Models, here, are conceived broadly, so as to include both representations internal to a thinker's mind – her memory of concepts or schemata – and external representations, models understood as cultural artifacts to be found in textbooks or laboratories. I depart from Grimm’s reference to “mental models” in order to consider a view as close to Grimm’s as possible that still remains non-committal on the cognitive activity that underwrites

would be that, when we ask how mathematical truths could have differed, we contemplate alternative models we use. My reply is that, while we may at times contemplate a variety of models, that isn't always required for understanding the phenomena theorized.

This text doesn't discuss a modeling approach to understanding *per se,* but only inasmuch as it offers a new interpretation of "What if things had been different?" questions asked in order to identify what a phenomenon depends on, so as to ultimately explain and understand that phenomenon.[15] A modeling approach wouldn't eliminate the role that "What if things had been

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understanding, for such psychological speculation would raise questions of plausibility on its own.

15 We should distinguish the view considered in this section from Saatsi's (2018) view. Saatsi writes that "genuine explanations are underwritten by explanatory dependencies in the world. This is the basic factivity requirement of the counterfactual-dependence account. Explanatory understanding, in turn, can be construed as an agent's ability to make correct counterfactual what-if inferences. [And] what matters for explanatory progress is that understanding-providing theories and models *de facto* latch onto reality in appropriate ways, so as to satisfy explanations' basic factivity requirement" (Saatsi 2018, pp. 14, 2). What Saatsi seems to be suggesting is that we explain phenomena by drawing inferences from the models we have of those phenomena. And models represent, not just what there is, but also how the target systems would behave in different conditions. This differs from the view attributed to Grimm here, on which "What if things had been different?" questions consider, not varying phenomena given a fixed model, but varying models of presumptively the same target phenomenon.

different?" questions play. Such questions are important in the natural and social sciences, where counterfactual scenarios are a good guide to testing. Rather, we answer "What if things had been different?" questions *differently*, by considering how our models could differ in how they represent the same target phenomenon, even when that phenomenon obtains necessarily.[16] The word "model" has an astonishing variety of uses.[17] However, in order to speak to the

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On a modeling approach to "What if things had been different?" questions, it doesn't matter that the antecedent of (b) may not pick out any metaphysically possible worlds. For we can ask: "What if things had been different – logically, set-theoretically, arithmetically, etc.?" And these questions would be neutral on metaphysical matters, showing that the third way of interpreting "What if things had been different?" questions is superior to the first two ways of interpreting them, sketched in Sections 3-4.

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After giving several quotes from science textbooks illustrating different uses of the word "model," Suppes writes: "The first of these quotations is taken from a book on mathematical logic, the next two from books on physics, the following three from works on the social sciences, and the last one from an article on mathematical statistics…One of the more prominent senses of the word missing in the above quotations is the very common use in physics and engineering of "model" to mean an actual physical model as, for example, in the phrases "model airplane" and "model ship" … It would, I think, be too much to claim that the word "model" is being used in exactly the same sense in all [these contexts]" (Suppes, 1960, p. 3). In arguing that "the meaning of the concept of model is the same in mathematics and the empirical sciences" (p. 4), Suppes is aware that he is going against first appearances.

counterexamples in Section 2, the view has to presumably consider models in the model-theoretical sense.

In the case of arithmetical understanding mentioned in Section 2, answering "What if things had been different?" questions has to meet two tasks. First, that we can make sense of the mathematical counterfactual (b). Second, that it is somehow *via* interpreting this counterfactual that we gain the understanding of the fact that 3 has a predecessor in (a). I believe that the first task is met but that the second is not.

Here are two models relative to which we may make sense of (b), or non-trivially assign it a truth value. First, consider a model of first-order predicate logic that has only two elements. This model would *not* also be a model of first-order Peano arithmetic. In this model, (b) is true because its antecedent is false. But the justification is not due to any metaphysical impossibility. Rather, the justification has to mention the *model* in which we interpret (b). Second, consider a model of two elements, but this time as a model of the positive integral domain $\mathbb{Z}_1$. On that model, for instance, $(4)_{\mathbb{Z}1}=(2)_{\mathbb{Z}1}=(0)_{\mathbb{Z}1}$ and $(5)_{\mathbb{Z}1}=(3)_{\mathbb{Z}1}=(1)_{\mathbb{Z}1}$. Mimicking the successor function for natural numbers, in $\mathbb{Z}_1$ the successor of 0 will be 1, the successor of 1 will be 0, and so on. "3" is interpreted as $(3)_{\mathbb{Z}1}$, and we can ascertain that it does have a predecessor. Again, (b) will come out true because its antecedent is false. But, again, the falsity of the antecedent of (b) and the truth of (b) are established partly with reference to the model in which we interpret (b).

I'll now argue that showing (b) is coherent, and can be interpreted non-trivially, doesn't *also* show that the understanding (a) delivers of why 3 has a predecessor has to be mediated by (b).

We might think as follows. We can test the validity of the argument in (a) by considering models, trying to falsify the conclusion and see if the premises are thereby falsified or not. Since the argument in (a) is valid, no such model will be forthcoming. And it is open to say, with Grimm, that we might better *appreciate* the validity of the argument in (a) by considering models. In considering such models, *volens nolens* we consider models that would give (b) the same truth value, albeit for different reasons. Whether we put the matter in terms of the verbalized subjunctive conditional (b) is irrelevant. What matters is that we entertain such models in testing if (a) is valid.

There is some plausibility to this line of thought, but it surely cannot hold in all cases, and we have no special reason for thinking it holds in the case of (a)-(b). The key point to notice is that we may have a purely syntactic understanding of why 3 has a predecessor. We grasp that all strictly positive natural numbers have predecessors, and that 3 is a strictly positive natural number; we grasp *modus ponens* and universal instantiation, and can thereby deduce the conclusion in (a), that 3 has a predecessor. We need appeal to no models in doing so.

The reply generalizes: we may have a purely *proof-theoretic* understanding of a conclusion without appealing to modeling. To be sure, valid arguments are such that, in any model where the premises are true, so is the conclusion. But mention of models is indirect, and it is unclear why understanding the conclusion of a proof *could not* be achieved without reference to models.

I am not advocating a proof-theoretical approach to mathematical understanding. Intuitively, the proof-theoretical approach and the model-theoretical approach are complementary, and precisely which of them (and in what contexts) better serves understanding should not be judged before the fact. But this is precisely what (3) does in implying that we can only understand the conclusion in (a) *via* (b).

In this section, I argued that a modeling approach to “What if things had been different?”

questions cannot rescue Grimm's view from counterexamples.[18]

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[18] To simplify things, the text focuses on the first counterexample to Grimm's view in Section 2. As for the second counterexample in Section 2, we might think that two images support the axiom of infinity. Either infinitely many points exist in spacetime (so infinitely many concrete individuals). Or we may build the iterative hierarchy of sets, which is as close as we may come to intuitiveness without running into inconsistency. (Field, 1980 and, respectively, Boolos, 1971 are the classical sources for these two images of sets.) Asking how things might have been different amounts to asking whether we may contemplate models that retain plausibility despite departing from these two images of what sets are. Finitism, for instance (even if *per impossibile*), is a live theoretical option. The point is not to definitively decide any issues with respect to infinity, but only to point to how basic theoretical and modeling choices inform the ways in which we understand the axiom of infinity. In reply, notice that calling the claim that there are infinitely many spacetime points, or the iterative theory of sets "images" plays no essential role. We might as well *deduce* the axiom of infinity from such claims and theories. Suppose we wrote the model theory for the Zermelo-Fränkel set theory in a meta-language of our choice, so that the "intuitive" iterative conception of sets is simply a theory in one's chosen meta-language, all of whose models are also models of the object-language Zermelo-Fränkel set theory being interpreted. All this is to say that no obvious benefit follows from appeal to "images," as opposed to just more theory.

Conclusion

On Grimm’s (2014) view, we only understand a phenomenon if we know what its occurrence depends on. On this view, we know what depends on what, and are enabled to explain target phenomena, by answering “What if things had been different?” questions. However, I developed two counterexamples to Grimm's view of understanding, and argued against three replies Grimm may make.

The fact that Grimm’s view meets with these counterexamples may suggest that different instances of understanding may vary significantly in their epistemic profile according to *what* is being understood: bits of mathematics, the occurrence of revolutions, or how the dosage of an anesthetic contributes to the success of an ensuing surgery. The variety of possible interpretations of “What if things had been different?” questions, and how we may answer them, reflects this. Whether an informative general, topic-neutral, epistemology of understanding does justice to the epistemic particularities of different instances of understanding is still an open question.

This dialectical situation is important because Grimm’s view considered understanding in full generality, whereas the counterexamples I mentioned are instances of mathematical understanding. The details of Grimm's view aside, it is worth emphasizing that the question of what mathematical dependencies ultimately consist in is left open, even though the fruitfulness of a purely metaphysical analysis of such dependencies may be undermined.

For a simple instance, think of "1+1=2".[19] This states an elementary fact, on which much of mathematics depends. In its turn, this fact depends on what identity and addition amount to, as well as on what the nature of numbers is. However, what might we mean in saying that some of these facts "depend on" one another? We might say that 1 is a *part of* 2. If numbers were sets, we might say that 1 is *an element of* 2. If we adjoined two collinear unities, we would thereby *construct* a segment that is two units long. We may build a *deductive proof* of "1+1=2" from Peano's axioms and the usual logical rules and axioms. Number-structures that include a position for 2 must include *substructures* that include a position for 1. And so on. We may cast such elementary dependencies in terms of deduction, construction, parthood, membership, etc. Which of these applies for developing an r understanding of arithmetical facts, and in what contexts, is, I

19 Alternatively, we may revert to the two counterexamples I have offered against Grimm's view. As for understanding that every strictly positive natural number has a predecessor, what is understood obviously depends on understanding what Peano's axioms express, together with logical operations and what the converse of a relation is (here, the converse of a successor-relation, viz., the associated predecessor-relation). As for the understanding the axiom of infinity, and the role that finite models may play in grasping the relations between set theory and arithmetic, see Krynicki and Zdanowski (2005). Since their discussion is more mathematically sophisticated, I chose to end with the much simpler "1+1=2" to illustrate questions surrounding what mathematical dependence consists in. I am indebted to an anonymous reviewer whose suggestions prompted me to rethink the conclusion of this paper.

believe, still an open question.[20] And whether these would count as distinct dependencies, or whether they merely *represent* in different ways what amounts to the same dependence of 2 on 1 is another question I leave open.

What *isn't* left open here is a methodological choice going forward. When we consider “What if things had been different?” questions as they apply to mathematics, or when we consider what mathematical dependencies may amount to, we should try to spell out the metaphysical or metamathematical views available, and discuss their specific details. I have here sketched Grimm's view, and argued it doesn't afford a notion of mathematical dependence correlative to “What if things had been different?” questions that may illuminate what our understanding of mathematical matters consists in.

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[20] For an elaboration of, and examples in support of, the view that understanding may depend, in mathematical contexts, on which specific problem we are in the process of solving, cf. Avigad (2008).

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